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TANNAKA-KREIN DUALITY FOR COMPACT GROUPOIDS III, DUALITY THEORY

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Abstract. In a series of papers, we have shown that from the representation theory of a compact groupoid one can reconstruct the groupoid using the procedure similar to the Tannaka-Krein duality for compact groups. In this part we introduce the Tannaka groupoid of a compact groupoid and show how to recover the original groupoid from its Tannaka groupoid.

1. introduction

This is the last in a series of papers in which we generalized the Tannaka-Krein duality to compact groupoids. In [A1] we studied the representation theory of compact groupoids. In particular, we showed that irreducible representations have finite dimensional fibres. We also proved the Schur's lemma, Gelfand-Raikov theorem and Peter-Weyl theorem for compact groupoids. In [A2] we studied the Fourier and Fourier-Plancherel transforms and their inverse transforms on compact groupoids. In this part we show how to recover a compact groupoid from its representation theory. This is done along the lines of the Tannaka duality for compact groups. We refer the interested reader to [JS] for a clear exposition of this theory. All over this paper we assume that $\mathcal{G}$ is compact and the Haar system on $\mathcal{G}$ is normalized. We put $X = \mathcal{G}^{(0)}$.

2. Tannaka groupoid

There is a forgetful functor $\mathcal{U} : \mathcal{R}ep(\mathcal{G}) \to \mathcal{H}il_X$ to the category of Hilbert bundles over X and operator bundles. A *natural transformation* $a : \mathcal{U} \to \mathcal{U}$ is a family of bundle maps $a_\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi$ indexed by $\mathcal{R}ep(\mathcal{G})$ such that for each $\pi_1, \pi_2 \in \mathcal{R}ep(\mathcal{G})$ and $h \in Mor(\pi_1, \pi_2)$ the following diagram commutes

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$$\begin{array}{ccc} \mathcal{H}_{\pi_1} & \xrightarrow{a_{\pi_1}} & \mathcal{H}_{\pi_1} \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathcal{H}_{\pi_2} & \xrightarrow[a_{\pi_2}]{} & \mathcal{H}_{\pi_2} \end{array}$$

One should understand this as each a_π being a bundle $a_\pi = \{a^\pi_{u,v}\}$ of bounded linear operators $a^\pi_{u,v} \in \mathcal{B}(H^\pi_u, \mathcal{H}^\pi_v)$ (possibly zero) indexed by $X \times X$ such that for each $u, v \in X$ the following diagrams commute

$$\begin{array}{ccc} \mathcal{H}^{\pi_1}_u & \xrightarrow{a^{\pi_1}_{u,v}} & \mathcal{H}^{\pi_1}_v \\ {\scriptstyle h_u}\big\downarrow & & \big\downarrow{\scriptstyle h_v} \\ \mathcal{H}^{\pi_2}_u & \xrightarrow[a^{\pi_2}_{u,v}]{} & \mathcal{H}^{\pi_2}_v \end{array}$$

Given $x \in \mathcal{G}$ there is a natural transformation $\mathcal{T}_x : \mathcal{U} \to \mathcal{U}$ defined by

$$(\mathcal{T}_x)^\pi_{u,v} = \begin{cases} \pi(x) & \text{if } u = s(x), v = r(x), \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Another interesting example of a natural transformation is the Fourier transform [A2]. Recall that we looked at $L^1(\mathcal{G})$ as a bundle of Banach algebras over $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ whose fiber at (u, v) is $L^1(\mathcal{G}^v_u, \lambda^v_u)$, and then each $f \in L^1(\mathcal{G})$ had its Fourier transform $\mathfrak{F}(f)$ in $C_0(\hat{\mathcal{G}}, \mathcal{B}(\mathcal{H}))$, where $\mathcal{B}(\mathcal{H})$ is a bundle of bundles of C^*-algebras over $\hat{\mathcal{G}}$ whose fiber at π is the bundle $\mathcal{B}(\mathcal{H}_\pi)$ over $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ whose fiber at (u, v) is $\mathcal{B}(\mathcal{H}^\pi_v, \mathcal{H}^\pi_u)$, the space $C_0(\hat{G}, \mathcal{B}(\mathcal{H}))$ is the set of all continuous sections vanishing at infinity, and $\mathfrak{F}(f)(\pi)_{(u,v)} = \mathfrak{F}_{u,v}(f_{(u,v)})(\pi)$. Now we need a flip in the order of u, v when we consider $\mathfrak{F}(f)$ as a natural transformation, namely we put $\mathfrak{F}(f)^\pi_{u,v} = \mathfrak{F}(f)(\pi)_{(v,u)}$. This way we get $\mathfrak{F}(f)^\pi_{u,v} \in \mathcal{B}(\mathcal{H}^\pi_u, \mathcal{H}^\pi_v)$. To see that this is indeed a natural transformation, note that for each $u, v \in X$, $x \in \mathcal{G}^u_v$, $\pi_1, \pi_2 \in \mathcal{R}ep(\mathcal{G})$, and $h \in Mor(\pi_1, \pi_2)$ we have

$$\begin{array}{ccc} \mathcal{H}^{\pi_1}_u & \xrightarrow{\pi_1(x^{-1})} & \mathcal{H}^{\pi_1}_v \\ {\scriptstyle h_u}\big\downarrow & & \big\downarrow{\scriptstyle h_v} \\ \mathcal{H}^{\pi_2}_v & \xrightarrow[\pi_2(x^{-1})]{} & \mathcal{H}^{\pi_2}_v \end{array}$$

Multiplying both sides with $f_{(v,u)}(x)$ and integrating against $d\lambda^u_v(x)$ we get

$$\begin{tikzcd}
\mathcal{H}_u^{\pi_1} \arrow[r, "\mathfrak{F}(f)^{\pi_1}_{u,v}"] \arrow[d, "h_u"'] & \mathcal{H}_v^{\pi_1} \arrow[d, "h_v"] \\
\mathcal{H}_v^{\pi_2} \arrow[r, "\mathfrak{F}(f)^{\pi_2}_{u,v}"'] & \mathcal{H}_v^{\pi_2}
\end{tikzcd}$$

which means $\mathfrak{F}(f) : \mathcal{U} \to \mathcal{U}$ is a natural transformation. Let $\mathcal{E}nd(\mathcal{U})$ be the set of all natural transformations $: \mathcal{U} \to \mathcal{U}$ with the coarsest topology making all maps $a \mapsto a^{\pi}_{u,v}$ continuous. Also we define an involution on $\mathcal{E}nd(\mathcal{U})$ by

$$\bar{a}^{\pi}_{u,v}(\xi) = \overline{a^{\bar{\pi}}_{u,v}(\bar{\xi})} \quad (u, v \in X, \pi \in \mathcal{R}ep(\mathcal{G}), \xi \in \overline{\mathcal{H}^{\pi}_u}).$$

The following is trivial.

Lemma 2.1. *$\mathcal{E}nd(\mathcal{U})$ is a topological vector space with continuous involution.* ☐

Proposition 2.2. *The map*

$$q : \mathcal{E}nd(\mathcal{U}) \to \prod_{\rho \in \hat{\mathcal{G}}} \mathcal{E}nd(\mathcal{H}_\rho)$$

$$a \mapsto (a_\rho)_{\rho \in \hat{\mathcal{G}}}$$

is an isomorphism of topological vector spaces.

Proof The following commutative diagrams (with vertical maps being canonical imbedding) illustrates that $a_{\pi_1 \oplus \pi_2} = a_{\pi_1} \oplus a_{\pi_2}$, for each $\pi_1, \pi_2 \in \mathcal{R}ep(\mathcal{G})$ and $a \in \mathcal{E}nd(\mathcal{U})$.

$$\begin{tikzcd}
\mathcal{H}_{\pi_1} \arrow[r, "a_{\pi_1}"] \arrow[d, "\iota_1"'] & \mathcal{H}_{\pi_1} \arrow[d, "\iota_1"] \\
\mathcal{H}_{\pi_1} \bigoplus \mathcal{H}_{\pi_2} \arrow[r, "a_{\pi_1 \oplus \pi_2}"'] & \mathcal{H}_{\pi_1} \bigoplus \mathcal{H}_{\pi_2} \\
\mathcal{H}_{\pi_2} \arrow[r, "a_{\pi_2}"'] \arrow[u, "\iota_2"] & \mathcal{H}_{\pi_2} \arrow[u, "\iota_2"']
\end{tikzcd}$$

This plus the fact that each representation of $\mathcal{G}$ is the direct sum of its irreducible sub representations [A1, theorem 2.16] shows that q is one-one. To show that it is onto, let $b = (b_\rho)$ with $b_\rho \in \mathcal{E}nd(\mathcal{H}_\rho)$ be given. Let $\pi \in \mathcal{R}ep(\mathcal{G})$ and $\mathcal{H}_\pi = \bigoplus_{\rho \in \hat{\mathcal{G}}} \mathcal{H}_{\pi_\rho}$ be the unique decomposition into isotropical components. For $\rho \in \hat{\mathcal{G}}$, the canonical map

$$\psi_\rho : \mathcal{H}_\rho \bigotimes Hom_{\mathcal{G}}(\mathcal{H}_\rho, \mathcal{H}_\rho) \to \mathcal{H}_\rho$$

$$\xi \otimes \varphi \mapsto \varphi(\xi)$$

is an isomorphism of $\mathcal{G}$-modules. Put $a_{\pi_\rho} = \psi_\rho \circ (b_\rho \otimes id) \circ \psi_\rho^{-1}$, $a_\pi = \bigoplus_{\rho\in\hat{\mathcal{G}}} a_{\pi_\rho}$, and $a = (a_\pi)_{\pi\in\mathcal{R}ep(\mathcal{G})}$. It is easy to see that $a : \mathcal{U} \to \mathcal{U}$ is a natural transformation and $a_\rho = b_\rho$, for each $\rho \in \hat{\mathcal{G}}$. Hence q is onto. The way we defined the topology of $\mathcal{E}nd(\mathcal{U})$ makes q^{-1} continuous. The fact that q is continuous is trivial. □

An alternative version of the above proposition would be to interpret q as a bundle isomorphism between bundles of bundles of C^*-algebras, that is $\mathcal{E}nd(\mathcal{U})$ is a bundle over $\hat{\mathcal{G}}$ whose fiber at $\rho \in \hat{\mathcal{G}}$ is a bundle of C^*-algebras over $X \times X$ whose fiber at (u, v) is the C^*-algebra $\mathcal{B}(\mathcal{H}_u^\pi, \mathcal{H}_v^\pi)$. This has the advantage of a better interpretation of the global Fourier transform. Indeed, in the light of [A2, Corollary 2.4], the above discussion could be rephrased as

Proposition 2.3. *The global Fourier transform*

$$\mathfrak{F} : L^1(\mathcal{G}) \to \mathcal{E}nd(\mathcal{U})$$

is a bundle homomorphism. □

Although $\mathcal{E}nd(\mathcal{U})$ doesn't seem to have a non trivial everywhere defined product, but one can define a "center" for it!

Definition 2.4. *The center $\mathcal{Z}(\mathcal{E}nd(\mathcal{U}))$ of $\mathcal{E}nd(\mathcal{U})$ consists of those $a \in \mathcal{E}nd(\mathcal{U})$ which commute with each $b \in \mathcal{E}nd(\mathcal{U})$ in the following sense*

$$b_{v,u}^\pi \circ a_{u,v}^\pi = a_{v,u}^\pi \circ b_{u,v}^\pi \quad (u, v \in X, \pi \in \mathcal{R}ep(\mathcal{G})).$$

Proposition 2.5. *$\mathcal{Z}(\mathcal{E}nd(\mathcal{U}))$ is a closed subspace of $\mathcal{E}nd(\mathcal{U})$ and the restriction of q gives an isomorphism*

$$\mathcal{Z}(\mathcal{E}nd(\mathcal{U})) \simeq \mathbb{C}^{\hat{\mathcal{G}}} = \prod_{\rho\in\hat{\mathcal{G}}} \mathbb{C}.id_\rho.$$

Proof The first statement is trivial. The second follows by a diagonalization argument. □

Using the notion of center, some of the results of the previous section on the Fourier transform of central functions could be rephrased in the terms of $\mathcal{E}nd(\mathcal{U})$. Here is one example.

Lemma 2.6. *Let $a \in \mathcal{E}nd(\mathcal{U})$, then $a \in \mathcal{Z}(\mathcal{E}nd(\mathcal{U}))$ if and only if there is $\pi \in \mathcal{R}ep(\mathcal{G})$ such that $a_\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi$ commutes with the action of $\mathcal{G}$, i.e.*

$$a_{v,u}^\pi \pi(x) = \pi(x^{-1}) a_{u,v}^\pi \quad (u, v \in X, x \in \mathcal{G}_u^v).$$

Proof If $a \in \mathcal{Z}(\mathcal{E}nd(\mathcal{U}))$, then for each $x \in \mathcal{G}$, a commutes with $\mathcal{T}^x$, so we have the above equality. Conversely, if this holds, then for each $\pi \in \mathcal{R}ep(\mathcal{G})$, $a_\pi \in Mor(\pi, \pi)$, so by the definition of the natural transformation, a commutes with each $b \in \mathcal{E}nd(\mathcal{U})$. □

Now Lemma 4.3 of [A2] could be rephrased as

Proposition 2.7. *If $f \in \mathfrak{C}C(\mathcal{G})$ then $\mathfrak{DF}(f) \in \mathcal{Z}(\mathcal{E}nd(\mathcal{U}))$.* □

Definition 2.8. *An element $a \in \mathcal{E}nd(\mathcal{U})$ is called monoidal (tensor preserving) if for each $\pi_1, \pi_2 \in \mathcal{R}ep(\mathcal{G})$, $a_{\pi_1 \otimes \pi_2} = a_{\pi_1} \otimes a_{\pi_2}$ and a_{tr} is trivial, i.e. for each $u, v \in X$ the following diagram commutes*

$$\begin{array}{ccc} \mathcal{H}_u^{\pi_1} \bigotimes \mathcal{H}_u^{\pi_2} & \xrightarrow{a_{u,v}^{\pi_1} \otimes a_{u,v}^{\pi_2}} & \mathcal{H}_v^{\pi_1} \bigotimes \mathcal{H}_v^{\pi_2} \\ \| & & \| \\ \mathcal{H}_u^{\pi_1 \otimes \pi_2} & \xrightarrow[a_{u,v}^{\pi_1 \otimes \pi_2}]{} & \mathcal{H}_v^{\pi_1 \otimes \pi_2} \end{array}$$

and $a_{u,v}^{tr} = id$, where tr is the trivial representation of $\mathcal{G}$ on $\mathbb{C}$. $a \in \mathcal{E}nd(\mathcal{U})$ is called Hermitian if $\bar{a} = a$.

Definition 2.9. *For each $u, v \in X$ and $a \in \mathcal{E}nd(\mathcal{U})$, consider the continuous section $a_{u,v}$ defined on $\mathcal{R}ep(\mathcal{G})$ by $a_{u,v}(\pi) = a_{u,v}^{\pi}$. The set $\mathcal{T}(\mathcal{G})$ of all sections $a_{u,v}$ where $a \in \mathcal{E}nd(\mathcal{U})$ is monoidal and Hermitian and $u, v \in X$ is called the Tannaka groupoid of $\mathcal{G}$. For fixed $u, v \in X$, we denote the set of all $a_{u,v} \in \mathcal{T}(\mathcal{G})$ by $\mathcal{T}_{u,v}(\mathcal{G})$.*

Theorem 2.10. *$\mathcal{T}(\mathcal{G})$ is a compact groupoid.*

Proof We define the product for the pairs of the form $(a_{w,v}, b_{u,w}) \in \mathcal{T}(\mathcal{G})^{(2)}$ by composition

$$(ab)_{u,v}^{\pi} = a_{w,v}^{\pi} \circ b_{u,w}^{\pi}.$$

This is clearly an associative partial operation on $\mathcal{T}(\mathcal{G})$.

It is easy to check that if $a, b \in \mathcal{E}nd(\mathcal{U})$ are monoidal and Hermitian, then so is ab. Indeed

$$\begin{aligned} (ab)_{u,v}^{\pi_1 \otimes \pi_2} &= a_{w,v}^{\pi_1 \otimes \pi_2} \circ b_{u,w}^{\pi_1 \otimes \pi_2} = (a_{w,v}^{\pi_1} \otimes a_{w,v}^{\pi_2}) \circ (b_{u,w}^{\pi_1} \otimes b_{u,w}^{\pi_2}) \\ &= (a_{w,v}^{\pi_1} \circ b_{u,w}^{\pi_1}) \otimes (a_{w,v}^{\pi_2} \circ b_{u,w}^{\pi_2}) = (ab)_{u,v}^{\pi_1} \otimes (ab)_{u,v}^{\pi_2}. \end{aligned}$$

For each $\pi \in \mathcal{R}ep(\mathcal{G})$ let $\check{\pi} \in \mathcal{R}ep(\mathcal{G})$ be its adjoint representation , and put

$$(a_{u,v}^{-1})^{\pi} := {}^t a_{u,v}^{\check{\pi}} \quad (u, v \in X, \pi \in \mathcal{R}ep(\mathcal{G})).$$

For each $u \in X$ define $\varepsilon_u : \mathcal{H}_u^{\check{\pi}} \bigotimes \mathcal{H}_u^{\pi} \to \mathbb{C}$ by

$$\varepsilon_u(\eta \otimes \xi) =< \eta, \xi > \qquad (\eta \in \mathcal{H}_u^{\check{\pi}} = (\mathcal{H}_u^{\pi})^*, \xi \in \mathcal{H}_u^{\pi}),$$

then we claim that $\varepsilon \in Mor(\check{\pi} \otimes \pi, tr)$. Indeed for each $x \in \mathcal{G}$ and $\eta \in \mathcal{H}_{s(x)}^{\check{\pi}}, \xi \in \mathcal{H}_{s(x)}^{\pi}$ we have

$$\begin{aligned} \varepsilon_{r(x)} \check{\pi} \otimes \pi(x)(\eta \otimes \xi) &= \varepsilon_{r(x)}(\check{\pi}(x)\eta \otimes \pi(x)\xi) =< \check{\pi}(x)\eta, \pi(x)\xi > \\ &=< {}^t\pi(x)^{-1}\eta, \pi(x)\xi >=< \eta, \xi > \\ &= \varepsilon_{s(x)}(\eta \otimes \xi) = tr(x)\varepsilon_{s(x)}(\eta \otimes \xi). \end{aligned}$$

Therefore for each $u, v \in X$ and $a \in \mathcal{E}nd(\mathcal{U})$ we have $\varepsilon_v a^{\check{\pi}\otimes\pi}_{u,v} = a^{tr}_{u,v}\varepsilon_u$. In particular for each $a_{u,v} \in \mathcal{T}(\mathcal{G})$, $\eta \in \mathcal{H}^{\check{\pi}}_u$, and $\xi \in \mathcal{H}^{\pi}_v$ we have

$$
\begin{aligned}
< a^{\check{\pi}}_{u,v}(\eta), a^{\pi}_{u,v}(\xi) > &= \varepsilon_v(a^{\check{\pi}}_{u,v}(\eta) \otimes a^{\pi}_{u,v}(\xi)) \\
&= \varepsilon_v(a^{\check{\pi}\otimes\pi}_{u,v}(\eta \otimes \xi)) = a^{tr}_{u,v}\varepsilon_u(\eta \otimes \xi) = < \eta, \xi > .
\end{aligned}
$$

Put $\eta = b^{\check{\pi}}_{v,w}(\zeta)$ with $\zeta \in \mathcal{H}^{\check{\pi}}_v$, then

$$< a^{\check{\pi}}_{u,v}(b^{\check{\pi}}_{v,w}(\zeta)), a^{\pi}_{u,v}(\xi) > = < b^{\check{\pi}}_{v,w}(\zeta), \xi >,$$

for each ζ, ξ as above. Hence, changing π to $\check{\pi}$, we get

$${}^t a^{\check{\pi}}_{u,v} \circ a^{\pi}_{u,v} \circ b^{\pi}_{v,w} = b^{\pi}_{v,w},$$

that is $a^{-1}_{v,u}a_{u,v}b_{v,w} = b_{v,w}$. Similarly $b_{w,v}a^{-1}_{v,u}a_{u,v} = b_{w,v}$. This shows that $\mathcal{T}(\mathcal{G})$ is a groupoid.

Next we show that $\mathcal{T}(\mathcal{G})$ is a closed subset of a compact groupoid. Recall that isotropy groups $\mathcal{G}^u_u$ are compact groups and the restriction of the invariant measure $d\lambda_u$ to $\mathcal{G}^u_u$ is a left (and so right) Haar measure. For each $\pi \in \mathcal{R}ep(\mathcal{G})$ and $u \in X$, let $g_u : \mathcal{H}^{\pi}_u \bigotimes \mathcal{H}^{\pi}_u \to \mathbb{C}$ be defined by

$$g_u(\xi, \eta) = \int_{G^u_u} < \pi(x)\xi, \eta > d\lambda^u_u(x) \quad (\xi, \eta \in \mathcal{H}^{\pi}_u).$$

Also define $h_u : \overline{\mathcal{H}^{\pi}_u} \bigotimes \mathcal{H}^{\pi}_u \to \mathbb{C}$ by $h_u(\xi, \eta) = g_u(\bar{\xi}, \eta)$. We claim that $h \in Mor(\bar{\pi} \otimes \pi, tr)$. Indeed for each $\xi, \eta \in \mathcal{H}^{\pi}_u$ and $x \in \mathcal{G}$ we have

$$
\begin{aligned}
h_{r(x)}(\bar{\pi} \otimes \pi)(x)(\xi \otimes \eta) &= h_{r(x)}(\bar{\pi}(x)\xi \otimes \pi(x)\eta) = g_{r(x)}(\pi(x)\bar{\xi} \otimes \pi(x)\eta) \\
&= \int < \pi(y)\pi(x)\bar{\xi}, \pi(x)\eta > d\lambda^{r(x)}_{r(x)}(y) \\
&= \int < \pi(x^{-1}yx)\bar{\xi}, \eta > d\lambda^{r(x)}_{r(x)}(y) \\
&= \int < \pi(y)\bar{\xi}, \eta > d\lambda^{s(x)}_{s(x)}(y) \\
&= g_{s(x)}(\bar{\xi} \otimes \eta) = h_{s(x)}tr(x)(\xi \otimes \eta).
\end{aligned}
$$

Therefore, for each $u, v \in X$ and $a \in \mathcal{E}nd(\mathcal{U})$ we have $h_v a^{\bar{\pi}\otimes\pi}_{u,v} = a^{tr}_{u,v}h_u$. In particular for each $a_{u,v} \in \mathcal{T}(\mathcal{G})$ using monoidal property we get $h_v(\overline{a^{\pi}_{u,v}(\xi)}, a^{\pi}_{u,v}(\eta)) = h_u(\bar{\xi}, \eta)$, that is $g_v(a^{\pi}_{u,v}(\xi), a^{\pi}_{u,v}(\eta)) = g_u(\xi, \eta)$, for each $\xi, \eta \in \mathcal{H}^{\pi}_u$. Now we can view g_u and g_v as new inner products on $\mathcal{H}^{\pi}_u$ and $\mathcal{H}^{\pi}_v$, respectively, and look at the unitary elements in $\mathbb{B}(\mathcal{H}^{\pi}_u, \mathcal{H}^{\pi}_v)$, then the above relation is just to say that $a^{\pi}_{u,v} \in \mathcal{U}(\mathcal{B}((\mathcal{H}^{\pi}_u, g_u), (\mathcal{H}^{\pi}_v, g_v)))$, whence

$$\mathcal{T}(\mathcal{G}) \subseteq \prod_{\pi,u,v} \mathcal{U}(\mathcal{B}((\mathcal{H}^{\pi}_u, g_u), (\mathcal{H}^{\pi}_v, g_v))),$$

a product of compact groupoids. The fact that $\mathcal{T}(\mathcal{G})$ is a closed subset of this groupoid follows immediately from the definition of the topology on $\mathcal{E}nd(\mathcal{U})$. $\square$

Now let's consider the natural transformations $\mathcal{T}_x \in \mathcal{E}nd(\mathcal{U})$, $x \in \mathcal{G}$. It is clear that for each $x \in \mathcal{G}$, $\mathcal{T}_x \in \mathcal{T}(\mathcal{G})$ and

$$\mathcal{T}_{xy} = \mathcal{T}_x \mathcal{T}_y \quad (x, y \in \mathcal{G}^{(2)}).$$

In particular the image of $\mathcal{G}$ under $\mathcal{T}$ is a subgroupoid of $\mathcal{T}(\mathcal{G})$. We identify $\mathcal{G}$ with its image in $\mathcal{T}(\mathcal{G})$. For each $u, v \in X$, let $\mathcal{T}_{u,v} : \mathcal{G}_u^v \to \mathcal{T}_{u,v}(\mathcal{G})$ be defined by $\mathcal{T}_{u,v}(x)(\pi) = \pi(x) \quad (x \in \mathcal{G}_u^v)$. Also we can define two adjoint maps

$$\mathcal{T}^* : \mathcal{R}ep(\mathcal{T}(\mathcal{G})) \to \mathcal{R}ep(\mathcal{G})$$

by

$$\mathcal{T}^*(\Pi)(x) = \Pi(\mathcal{T}_x) \quad (x \in \mathcal{G}, \Pi \in \mathcal{R}ep(\mathcal{T}(\mathcal{G}))),$$

and

$$\mathcal{T}^* : \mathcal{E}(\mathcal{T}(\mathcal{G})) \to \mathcal{E}(\mathcal{G})$$

by

$$\mathcal{T}^*(f)(x) = f(\mathcal{T}_x) \quad (x \in \mathcal{G}, f \in \mathcal{E}(\mathcal{T}(\mathcal{G}))).$$

Lemma 2.11. *The restriction map*

$$\mathcal{T}^* : \mathcal{R}ep(\mathcal{T}(\mathcal{G})) \to \mathcal{R}ep(\mathcal{G})$$

is a bundle isomorphism.

Proof We define the extension bundle map $\mathfrak{E} : \mathcal{R}ep(\mathcal{G}) \to \mathcal{R}ep(\mathcal{T}(\mathcal{G}))$ as follows. Given $u, v \in X$, $a_{u,v} \in \mathcal{T}(\mathcal{G})$, and $\pi \in \mathcal{R}ep(\mathcal{G})$, the map $P_\pi : a_{u,v} \mapsto a^\pi_{u,v}$ is a representation of $\mathcal{T}(\mathcal{G})$ on $\mathcal{H}_\pi$ and we have the commutative triangles

$$\begin{array}{ccc} \mathcal{G}_{u,v} & \xrightarrow{\pi} & \mathcal{B}(\mathcal{H}^\pi_u, \mathcal{H}^\pi_v) \\ {\scriptstyle \mathcal{T}_{u,v}}\downarrow & & \uparrow{\scriptstyle P_\pi} \\ \mathcal{T}_{u,v}(\mathcal{G}) & \xrightarrow[=]{} & \mathcal{T}_{u,v}(\mathcal{G}) \end{array}$$

Therefore $\mathcal{T}^*(P_\pi) = \pi$. We put $\mathfrak{E}(\pi) = P_\pi$. If $h \in Mor_{\mathcal{G}}(\pi_1, \pi_2)$ then clearly $h \in Mor_{\mathcal{T}(\mathcal{G})}(P_{\pi_1}, P_{\pi_2})$. Also it is easy to check that $\mathfrak{E}$ preserves direct sums, tensor products, and conjugation of representations. Moreover the above commutative triangle shows that if π is irreducible, then so is (π). Hence $Im(\mathfrak{E})$ is a closed subset of $\mathcal{T}(\mathcal{G})$ in the sense of Definition 3.7 of [A2]. It also separates the points of $\mathcal{T}(\mathcal{G})$. Indeed If $a_{u,v}$ and $b_{w,z}$ are distinct elements of $\mathcal{T}(\mathcal{G})$, there is a representation $\pi \in \mathcal{R}ep(\mathcal{G})$ such that $a^\pi_{u,v} \neq b^\pi_{w,z}$, which means that P_π separates $a_{u,v}$ and $b_{w,z}$. By [A2, proposition 3.8], $\mathfrak{E}$ is surjective. Now $\mathcal{T}^* \circ \mathfrak{E} = id$, so $\mathcal{T}^*$ is a bundle isomorphism. □

Now let $\mathcal{E}(\mathcal{G})$ and $\mathcal{E}(\mathcal{T}(\mathcal{G}))$ be the representation bundles of $\mathcal{G}$ and $\mathcal{T}(\mathcal{G})$, respectively.

Lemma 2.12. *The restriction map*

$$\mathcal{T}^* : \mathcal{R}ep(\mathcal{T}(\mathcal{G})) \to \mathcal{R}ep(\mathcal{G})$$

is a bundle isomorphism.

Proof We define the extension bundle map $\mathfrak{E} : \mathcal{R}ep(\mathcal{G}) \to \mathcal{R}ep(\mathcal{T}(\mathcal{G}))$ as follows. Given $u, v \in X$, by Proposition 2.2 of [A2], any $f \in \mathcal{E}^\pi_{u,v}$ has a unique representation in the form

$$f = \sum_{\pi\in\hat{\mathcal{G}}} d^\pi_u Tr\big(g(\pi))\pi(.)\big),$$

where $g = \mathfrak{F}_{u,v}(f) \in \sum_{\pi\in\hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}^\pi_v, \mathcal{H}^\pi_u)$. Define $\mathfrak{E}_{u,v}(f)$ on $\mathcal{T}_{u,v}(\mathcal{G})$ by

$$\mathfrak{E}_{u,v}(f)(a_{u,v}) = \sum_{\pi\in\hat{\mathcal{G}}} d^\pi_u Tr\big(g(\pi))P_\pi(a_{u,v})\big) \quad (a_{u,v} \in \mathcal{T}_{u,v}(\mathcal{G})).$$

By above lemma, $\mathfrak{E}$ is injective, so $\mathcal{T}^*$ is bijective, as we have $\mathcal{T}^* \circ \mathfrak{E} = id$. □

Lemma 2.13. *For each $f \in C(\mathcal{T}(\mathcal{G}))$ and $u, v \in X$,*

$$\int_{\mathcal{T}(\mathcal{G})^v_u} f(t)d\tilde{\lambda}^v_u(t) = \int_{\mathcal{G}^v_u} f(\pi(x))d\lambda^v_u(x).$$

Proof By Lemma 3.9 of [A1], it is enough to prove this for $f \in \mathcal{E}(\mathcal{T}(\mathcal{G}))$. As in the proof of the above lemma we may represent f on $\mathcal{T}(\mathcal{G})^v_u$ as

$$f(t) = \sum_{\pi\in\hat{\mathcal{G}}} d^\pi_u Tr\big(g(\pi))P_\pi(t)\big) \quad (t \in \mathcal{T}(\mathcal{G})^v_u),$$

where $g = \mathfrak{F}(\mathcal{T}^*(f))$. In particular, for each $x \in \mathcal{G}^v_u$,

$$f(\mathcal{T}_x)) = \sum_{\pi\in\hat{\mathcal{G}}} d^\pi_u Tr\big(g(\pi))P_\pi(\mathcal{T}_x)\big) = \sum_{\pi\in\hat{\mathcal{G}}} d^\pi_u Tr\big(g(\pi))\pi(x)\big).$$

By Proposition 3.2 (iii) of [A2], we have

$$\int Tr\big(g(\pi)\pi(x)\big)d\lambda^v_u(x) = \begin{cases} g(tr) & \text{if } \pi = tr, \\ 0 & \text{otherwise,} \end{cases}$$

where tr is the trivial representation , and similarly

$$\int Tr\big(g(\pi)P_\pi(t)\big)d\tilde{\lambda}^v_u(t) = \begin{cases} g(tr) & \text{if } \pi = tr, \\ 0 & \text{otherwise,} \end{cases}$$

hence the result. □

Now we are ready to prove the main result of these series of papers, the *Tannaka-Krein duality theorem* for compact groupoids.

Theorem 2.14. (Tannaka-Krein Duality Theorem) *Any compact groupoid is isomorphic to its Tannaka groupoid.*

Proof Let $\mathcal{G}$ be a compact groupoid. We show that $\mathcal{T} : \mathcal{G} \to \mathcal{T}(\mathcal{G})$ is an isomorphism of topological groupoids. The injectivity of $\mathcal{T}$ follows from the Peter-Weyl theorem [A1, theorem 3.13]. For the surjectivity, assume on the contrary that $Im(\mathcal{T})$ is a proper subset of $\mathcal{T}(\mathcal{G})$. This is a closed subset. Let $f \in C(\mathcal{T}(\mathcal{G}))$ be a positive function such that $supp(f)$ is contained in the complement of $Im(\mathcal{T})$. Then from the two integrals in above lemma, the one on the right hand side is 0, where as the one on the left hand side is strictly positive, a contradiction. □

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